Statistical Mechanics

Exam

3/4-2017

For this exam no books, tablets, or smartphones are allowed.
Location: 5161.0151, Time: 9:00-12:00.

Norm:

The answers must be given in English.
The table below shows the number of points to be given for each of the questions. For the exam score the total score, M, is converted to the mark using the max score (M_{max}=54) according to the formula $\frac{9M}{M_{\text{max}}} + 1$. If 1/2 bonus point was earned during the problem solving session this is added and if the resulting mark is above 10 the final mark will be 10.

Subquestion	Q1	Q2	Q3	Q4
A	2	4	4	2
B	4	2	2	4
C	4	4	2	4
D	2	4	4	2
E	-	-	-	4

1 The Ideal Gas

Consider two ideal gasses each consisting of N molecules contained in their own volume V at identical temperature T. The internal energy of each gas is U_1 and U_2 and the masses are m_1 and m_2, respectively.
A. What is the entropy of each individual gas?
B. Derive the expression for the change in entropy when the two gasses are mixed assuming that they are different.
C. Derive the expression for the change in entropy when the two gasses are mixed assuming that they are identical.
D. Explain the difference between the result of question B and C.

2 The quantum harmonic oscillator

Consider an ensemble of independent one-dimensional quantum mechanical harmonic oscillators with the energy of each quantum level given by $E_n = \hbar\omega(n + 1/2)$. They all have the same fundamental frequency, ω.
A. Derive the expression for the partition function for one independent quantum mechanical harmonic oscillators.
B. Now consider a collection of N non-interacting harmonic oscillators. Derive the partition function.
C. Derive the Helmholtz free energy for the system.
D. Derive the entropy of the system of non-interacting quantum mechanical harmonic oscillators.

3 Phase Transitions

A. Consider a system described with the grand partition function

$$\mathcal{Q}(z, V) = (1+z)^V(1+z^{\alpha V})(1+2z^{\beta V}), \tag{1}$$

where α and β are positive constants, z is the fugacity and V the volume. Write down the equation of state for this system (present equations for P/kT and $1/v$).
B. What are the roots of this grand partition function?
C. In which limit will this system exhibit a phase transitions and for which values of the fugacity does it happen?
D. What is the order of the phase transition(s)? Motivate your answer.

4 Fermi systems

Consider an ideal Fermi gas, with spin 1/2 particles. All particles have the same mass, m.
A. Give the internal energy for the gas at low temperature. The expression should explicitly give the dependence on the degeneracy.
B. Use the Helmholtz energy of the Fermi gas to derive the pressure of the Fermi gas at low temperature. The Helmholtz energy is given as:

$$A_{1/2} = \frac{3}{5}N\epsilon_F\left(1 - \frac{5\pi^2}{12}\left(\frac{k_BT}{\epsilon_F}\right)^2 + \cdots\right) \tag{2}$$

What is the pressure of the Fermi gas?
C. Consider two containers, one with the spin 1/2 gas and one with a classical ideal gas connected with a freely moving piston. The number of particles, N, in the two containers is the same. Write the equation determining the volume of the Fermi gas, V_{Fermi}, given that the total volume is V_{Total}. (You do not need to isolate V_{Fermi} in the equation, but it may not explicitly depend on the volume of the classical ideal gas.)
D. What happens if the temperature goes to zero?
E. Now consider two containers, one with the spin 1/2 gas and one with a classical van der Waals gas connected with a freely moving piston. The number of particles, N, in the two containers is the same and the temperature is 0 K. What is the volume of the Fermi gas, V_{Fermi}, given that the

total volume is V_{Total}?

This exam has been drafted by T. L. C. Jansen and verified by T. A. Schlathölter.

Date: 24/3-2017
Signature T. L. C. Jansen:

Date: 24/3-2017
Signature T. A. Schlathölter:

Potentially useful equations

Maxwell relations:

$$
\begin{aligned}
P &= -\left(\frac{\partial A}{\partial V}\right)_T \\
P &= -\left(\frac{\partial U}{\partial V}\right)_S \\
V &= \left(\frac{\partial H}{\partial P}\right)_S \\
V &= \left(\frac{\partial G}{\partial P}\right)_T \\
S &= -\left(\frac{\partial A}{\partial T}\right)_V \\
S &= -\left(\frac{\partial G}{\partial T}\right)_P \\
T &= \left(\frac{\partial H}{\partial S}\right)_P \\
T &= \left(\frac{\partial U}{\partial S}\right)_V \\
\mu &= \left(\frac{\partial A}{\partial N}\right)_{V,T} \\
\mu &= \left(\frac{\partial G}{\partial N}\right)_{P,T}
\end{aligned}
$$

The thermal wavelength:

$$\lambda = \sqrt{2\pi\hbar^2/mkT}$$

The Ideal gas law:

$$PV = nRT \tag{3}$$

The van der Waas equation:

$$\left[P + a\left(\frac{n}{V}\right)^2\right]\left(\frac{V}{n} - b\right) = RT$$

The Sacktur-Tetrode equation:

$$S = k_B N\left(\log\left[\frac{V}{N}\left(\frac{4\pi m}{3h^2}\frac{U}{N}\right)^{3/2}\right] + \frac{5}{2}\right)$$

The functions for the ideal Fermi gas including the low-fugacity expansions:

$$
\begin{aligned}
f_{5/2}(z) &= \sum_{l=1}^{\infty} \frac{(-1)^{l+1} z^l}{l^{5/2}} \\
f_{3/2}(z) &= \sum_{l=1}^{\infty} \frac{(-1)^{l+1} z^l}{l^{3/2}} \approx z - \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} - \frac{z^4}{4^{3/2}} + \cdots
\end{aligned}
$$

$$
z = \frac{\lambda^3}{v} + \frac{1}{2^{3/2}} \left(\frac{\lambda^3}{v} \right)^2 + \cdots
$$

$$
\epsilon_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2}{gv} \right)^{2/3}
$$

$$
U = \frac{3}{5} N \epsilon_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{kT}{\epsilon_F} \right)^2 + \cdots \right]
$$

The functions for the ideal Bose gas including the low-fugacity expansions

$$
\begin{aligned}
g_{5/2}(z) &= \sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}} \\
g_{3/2}(z) &= \sum_{l=1}^{\infty} \frac{z^l}{l^{3/2}} \approx z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \frac{z^4}{4^{3/2}} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
g_{5/2}(1) &= \zeta(5/2) = 1.342 \cdots \\
g_{3/2}(1) &= \zeta(3/2) = 2.612 \cdots
\end{aligned}
$$

Scaling laws
Fisher: $\gamma = \nu(2 - \eta)$
Rushbrooke: $\alpha + 2\beta + \gamma = 2$
Widom: $\gamma = \beta(\delta - 1)$
Josephson: $\nu d = 2 - \alpha$

5 Mathematical relations

Geometric series valid for $x < 1$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\exp(-ax) = \sum_{n=0}^{\infty} \frac{(-ax)^n}{n!}$$

With log() the natural logarithm is considered. For $-1 < x < 1$:

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-x)^{n+1}}{n}$$

Potentially useful constants

$m_e = 9.10938356 \times 10^{-31}$ kg
$N_A = 6.022140857 \times 10^{23}$ 1/mol
$\hbar = 1.0545718 \times 10^{-34}$ Js
$k_B = 1.38064852 \times 10^{23}$ J/K
$c = 2.99792458 \times 10^{8}$ m/s